

Phase transition in ferromagnetic Ising model with a cell-board external field.

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October 22, 2015

Abstract

We show the presence of a first-order phase transition for a ferromagnetic Ising model on \mathbb{Z}^2 with a periodical external magnetic field. The external field takes two values h and $-h$, where $h > 0$. The sites associated with positive and negative values of external field form a cell-board configuration with rectangular cells of sides $L_1 \times L_2$ sites, such that the total value of the external field is zero. The phase transition holds if $h < \frac{2J}{L_1} + \frac{2J}{L_2}$, where J is an interaction constant. We prove the first-order phase transition using the reflection positivity (RP) method. We apply a key inequality which is usually referred to as the chessboard estimate.

Keywords: *Ising model, periodic external field, Peierls condition, reflection positivity, phase transition.*

1 Introduction

In many models of statistical physics the phase transition is a result of spontaneous breaking of the symmetry of a system. The best known model with phase transition is the ferromagnetic Ising model (system) in the absence of a magnetic field. Essentially, this fact has been shown by Peierls [24]. It has become a theorem by Griffiths [20] and Dobrushin [11] (see also [27] and [10]). Peierls ideas are referred to as *Peierls arguments* based on *Peierls condition* and *Peierls transformation*. Peierls condition means that energy required for a droplet formation of one of the phases surrounded by the sites occupied by another phase is proportional to the size of the droplet boundary. For a two dimensional model (on \mathbb{Z}^2) the boundary size is the length of the droplet's boundary. The second component of Peierls arguments allows to perform Peierls transformation. It is based on a symmetry which a studied model has. By Peierls transformation, it is possible to remove a contour in the configuration such that only the energy of the contour is eliminated, and the energy of the rest part of the configuration is not changed. Peierls condition is unrelated to the model symmetry. Peierls condition is satisfied for the Ising model with a uniform external field, however there is no the symmetry in this case.

Peierls arguments show a type of “stability” of ground states. It means that at a low temperature the state is ensemble of small perturbations of the ground state which would result in a configuration “close” to the starting ground state.

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Unlike Peierls argument (specifically Peierls transformation), the *Pirogov-Sinai theory* of phase transitions allows one to find a low-temperature phase diagram of models with no symmetry requirement. When there is no a symmetry, the low-temperature phase diagram is shifted with respect to the ground state diagram.

In addition, there exist a several more approaches. One such approach, *Reflection Positivity* (RP), requires showing a type of reflection symmetry. Essentially, it is possible to prove a phase transition constructing a contour argument using the *chessboard inequality* obtained from the RP property.

An external field added to the Hamiltonian can change the whole phase diagram. In the case of the ferromagnetic Ising model, any non-zero uniform external field suppresses the phase transition. In some models where the magnetic field is not supposed to be uniform, it is possible to prove phase uniqueness, see for instance, [6], [7]. A random external field can also suppress the phase transition in a planar Ising model (see [1], [2]), even in the case when the total average of the external field is equal to 0.

In this paper we will address the problem of the existence of phase transitions in a planar Ising model where the external field is periodic, forming a cell-board configuration such that total value of the magnetic field is zero. The initial motivation is coming from image processing where Ising models with non-uniform external fields are used for analysing segmentation. The model in this study firstly were numerically studied by M. Sigelle in [26]. Reviews on the applications of Gibbs fields in image processing can be found in Descombes and Zhizhina [9] (see also [22] and the book [28]). Posteriorly, Darbon and Sigelle [8] proposed a grayscale fast and exact optimization method, it decomposing the target image in layers behaving as Ising models with cell-board external fields.

The models with staggered external fields can be useful in the theory of surfaces and domain theory of the solids.

In this work we consider the Ising model where the external field takes two values h and $-h$, where $h > 0$. The lattice \mathbb{Z}^2 is split into the union of disjointed cells of the same size, and the signs of the external field are alternated similar to a chessboard. Specifically, the cell with one sign of the external field is surrounded by four neighbor cells with the opposite value of the external field. We propose the reflection positivity method for the studies of this model. We will use a specific term for the alternated external field, *cell-board* partition, to avoid a confusion with the chessboard estimate, to be used later in the paper.

In [23] (by F.R. Nardi, E. Olivieri, and M. Zahradník) the authors study the case when the cells have infinite horizontal length, while their height equals one. Except the phase coexistence at low temperature, a lot of effort in [23] are focused on the proof of the uniqueness in a parameter region where the ground states coexist.

Since our work is concentrated to the cases where Peierls condition is fulfilled then Pirogov-Sinai theory can be applied in this case. However, we used the reflection positivity method based on the periodicity of the cell-board external field. In this sense, as we will state in Corollary 2.4, a particular case of cell-board models when the size of the cells is 1×1 , is trivially related with antiferromagnetic Ising model with uniform external field (see [13]). For that model, in [16] the RP method has been used to prove the phase transition showed by Dobrushin [12]. Also RP property has been used to prove phase transition in planar rotor models with staggered external field (van Enter and Ruszel [14]). In addition, Fröhlich et al. [15] claimed that RP methods would produce better bounds of the critical temperature than Pirogov-Sinai approach can propose.

The paper is organized as follows: In sect. 2 we define the model we study, and present our main result (Theorem 2.2). Sect. 3 contains a brief description of main ideas of the proof. The *reflection positivity* technique, which is the main tool we use for the proofs, will be discussed in sect. 4.1, where we will also describe the *chessboard estimates*. The proof of the main result using the RP technique follows the standard scheme, (see for example [4], Chapters 5, 6) and are given in sect. 4.2. The chessboard estimates as constructed in sect. 4.1 does not encompass the external field in [23], therefore in sect. 5 we study using RP a generalization of that model, again in the region with Peierls condition.

2 Definitions and results

We study the ferromagnetic Ising model on \mathbb{Z}^2 with a periodic external field introduced in [26] (see also [22]). Represent the lattice \mathbb{Z}^2 as the union of rectangular cells of the size $L_1 \times L_2$, $L_i \in \mathbb{N}$: for each pair of integers n, m we define

$$C(n, m) = \{(t_1, t_2) \in \mathbb{Z}^2 : nL_1 \leq t_1 < (n+1)L_1, \\ mL_2 \leq t_2 < (m+1)L_2\}. \quad (2.1)$$

That is $\mathbb{Z}^2 = \cup_{n,m \in \mathbb{Z}} C(n, m)$. Let us define subsets \mathbf{Z}_+ and \mathbf{Z}_- of \mathbb{Z}^2 :

$$\mathbf{Z}_+ = \bigcup_{\substack{n,m: \\ n+m \text{ is even}}} C(n, m), \quad \mathbf{Z}_- = \mathbb{Z}^2 \setminus \mathbf{Z}_+. \quad (2.2)$$

A site of \mathbb{Z}^2 is colored white if it is from \mathbf{Z}_+ and black otherwise. Thus, the whole lattice is like a chessboard (see Figure 2(a), where $L_1 = 3$ and $L_2 = 2$).

Further we use a term *cell-board* since the term chessboard is used by reflection positivity technics which we apply.

Let $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$ be the set of all configurations on \mathbb{Z}^2 . The formal Hamiltonian is defined by

$$H(\sigma) = -J \sum_{\langle t, s \rangle} \sigma(t)\sigma(s) - \sum_s h(s)\sigma(s), \quad (2.3)$$

for any $\sigma \in \Omega$, where $\sigma(t) \in \{-1, +1\}$ is a *spin value* of configuration σ at the site $t \in \mathbb{Z}^2$, $J > 0$ is an interaction constant, a symbol $\langle t, s \rangle$ denotes unordered pairs of nearest neighbors $s, t \in \mathbb{Z}^2$, that is the Euclidean distance between the sites is one, $|t - s| = 1$, and an external field h is given by

$$h(s) = \begin{cases} h, & \text{if } s \in \mathbf{Z}_+, \\ -h, & \text{if } s \in \mathbf{Z}_-. \end{cases} \quad (2.4)$$

Further, for any subset $\Lambda \subset \mathbb{Z}^2$ and any configuration $\sigma \in \Omega$, we will use the notation $\sigma(\Lambda)$ for the configuration of σ restricted to the set of sites Λ .

We recall the standard definitions of a Gibbs field on the infinite lattice \mathbb{Z}^2 and related notations. Let W be a finite subset from \mathbb{Z}^2 , and let Ω_W be the set of all configurations on W : $\Omega_W = \{-1, 1\}^W$. The Gibbs probability of the configuration $\sigma \in \Omega_W$ with boundary conditions $\omega \in \Omega$, is given by

$$\mu_{\beta,W}(\sigma|\omega) = \frac{1}{Z_W(\beta)} \exp\left(\beta J \sum_{\substack{\langle t,s \rangle: \\ t,s \in W}} \sigma(t)\sigma(s) + \beta J \sum_{\substack{\langle t,s \rangle: \\ t \in W, s \notin W}} \sigma(t)\omega(s) + \beta \sum_{s \in W} h(s)\sigma(s)\right), \quad (2.5)$$

where β is a positive constant usually interpreted as the inverse temperature, and $Z_W(\beta)$ is a normalizing constant, called a *partition function*.

Let \mathcal{G}_β be a set of all Gibbs states on Ω obtained by the thermodynamic limit.

A configuration $\tilde{\sigma} \in \Omega$ is a local perturbation of a configuration $\sigma \in \Omega$ if there exists a finite set $V \subset \mathbb{Z}^2$ such that

$$\tilde{\sigma}(t) = \begin{cases} -\sigma(t), & \text{if } t \in V, \\ \sigma(t), & \text{if } t \notin V. \end{cases} \quad (2.6)$$

A configuration $\sigma \in \Omega$ is called a *ground state* for the Hamiltonian H , if for any local perturbation $\tilde{\sigma}$ of the configuration σ the inequality

$$H(\tilde{\sigma}) - H(\sigma) \geq 0,$$

is valid. Following [27] we say that the *Peierls condition* holds true, if there exists a positive constant $c_P > 0$ such that for any local perturbation $\tilde{\sigma}$ (as in (2.6)) of a ground state σ the inequality

$$H(\tilde{\sigma}) - H(\sigma) \geq c_P |\partial V|, \quad (2.7)$$

holds, where $\partial V = \{\langle t, s \rangle : t \in V, s \notin V\}$ is the boundary of the set V . The constant c_P is called the *Peierls constant*.

The following theorem provides the known results from [22] about the ground states and the Peierls condition for our model.

Theorem 2.1. *If*

$$h < \frac{2J}{L_1} + \frac{2J}{L_2}, \quad (2.8)$$

then there exist two periodical ground states, namely the constant configurations $\sigma^+ \equiv +1$ and $\sigma^- \equiv -1$. In addition, the Peierls condition holds, and the Peierls constant c_P is equal to $2J - hL_1L_2/(L_1 + L_2)$. If (2.8) does not hold and

$$h > \frac{2J}{L_1} + \frac{2J}{L_2}, \quad (2.9)$$

then the configuration

$$\sigma_c(t) = \begin{cases} +1, & \text{if } t \in \mathbf{Z}_+, \\ -1, & \text{if } t \in \mathbf{Z}_-, \end{cases} \quad (2.10)$$

is the unique periodic ground state.

2.1 Main result. Phase transition for cell-board model

The next theorem provides the presence of a first-order phase transition in the cell-board model.

Theorem 2.2. *Let the condition (2.8) hold true, then there exists some $\beta_0 = \beta_0(L_1, L_2)$, such that for any $\beta > \beta_0$, there exist two distinct measures μ_β^+ and $\mu_\beta^- \in \mathcal{G}_\beta$, which satisfy*

$$\mu_\beta^\pm(\sigma(t) = \pm 1) > \frac{1}{2}. \quad (2.11)$$

That means $|\mathcal{G}_\beta| > 1$. Moreover

$$\beta_0 = \frac{8[(B_1 B_2 + 4) \ln 2 + \ln(c(c+1))]}{2J - \frac{h L_1 L_2}{L_1 + L_2}}, \quad (2.12)$$

where B_i , $i = 1, 2$ are defined in (4.13) and $c > 1$ is a combinatorial constant related to the number of contours of a given size.

Remark 2.3. :

- Estimates for c can be found in [21] and [3]. In our case c can be taken no greater than 9.
- The denominator in (2.12) is the Peierls constant defined in Theorem 2.1.
- Let β_c the inverse critical temperature, then $\beta_0 \geq \beta_c$.

Theorem 2.2 is the main result in the paper. The proof is in sect. 4.2. It is based on the reflection positivity machinery. We explain the reflection positivity (RP) technique in a way adapted to our model in the section 4.

We conclude this section with a well known fact about connection between a particular case of our model and the antiferromagnetic Ising model with a constant external field. The formal Hamiltonian for the antiferromagnetic model is

$$H_a(\sigma) = -J_a \sum_{\langle t, s \rangle} \sigma(t) \sigma(s) - h_a \sum_s \sigma(s), \quad (2.13)$$

where the interaction constant J_a is negative, $J_a < 0$, that creates the antiferromagnetic interactions between the nearest spins, the external field h_a is a real constant. The external field, $\pm h$, of the cell-board model, when $L_1 = L_2 = 1$, should be equivalent to the antiferromagnetic model with the constant external field $h_a = h$. This fact has been discussed by Frohlich et al. [16] in the context of RP applications (see also [13]). In our settings this result is the consequence of Theorem 2.2.

Corollary 2.4. *Let $J, h > 0$. If $h < 4J$ and $\beta > \frac{2k}{4J-h}$ for some $k > 0$, then the antiferromagnetic Ising model (2.13) with $J_a = -J$ and $h_a = h$ has two phases.*

Proof. Consider the cell-board model with $L_1 = L_2 = 1$, as defined in (2.3) and (2.4). Now, define the transformation Ψ of the configuration space Ω , $\Psi : \Omega \rightarrow \Omega$,

$$\Psi(\sigma)(t_1, t_2) = \begin{cases} \sigma(t_1, t_2), & \text{if } t_1 + t_2 \text{ even,} \\ -\sigma(t_1, t_2), & \text{otherwise,} \end{cases}$$

is an one-to-one transformation of Ω . Note that if in (2.13) we choose $J_a = -J$, where $J > 0$, and $h_a = h$, then the transformation Ψ does not change energy of the configurations and provides the direct equivalence of the models. \square

3 Plan of the proof of Theorem 2.2

Our model has a set of reflection symmetries that allow us to apply the reflection positivity technique (see subsection 4.1). The proof of the RP property is Proposition 4.3. The reflections are with respect to lines parallel to the coordinate axes. Depending on the parity of sides L_1 and L_2 , the reflecting lines can either go through the sites of \mathbb{Z}^2 or bisect edges of \mathbb{Z}^2 . In our model, not all such lines are reflecting. As a result there are blocks of the sites in \mathbb{Z}^2 which do not have the reflection property, those blocks entirely reflected with respect to the reflecting lines. Definitions of the blocks see in (4.11) and (4.12).

We take a torus as a main scene of our considerations. The thermodynamical limit is corresponding to the growth of the torus size.

Proving the main Theorem 2.2 we estimate the probability to have different spin values $+1$ and -1 at remote sites on the torus (see Proposition 4.5). The goal is to show that this probability is small. It is clear that the event

$$(\sigma(s) = +1, \sigma(t) = -1),$$

when $s \neq t$, should generate Peierls contour which is a set of the edges having the different values on the edge ends. We use the contour arguments for the proof, however we have to use *thick contours* (*block contours*) consisted of the blocks. The block contour is composed of the blocks in which Peierls contour is passing. Any configuration on each such block takes the different spin values (a *bad block*). There is an exclusion which should be treated separately (see about *double-blocks* in section 4.2). A small probability of the configurations on the bad block follows from the chessboard estimate (Theorem 4.2) and from the Peierls condition (Lemma 4.7, see also Proposition 4.4). The chessboard estimate is applied to find an upper bound of the bad block probability, see (4.35). The Peierls condition is applied to make this upper bound small at small temperature.

4 A detailed plan of the proof. Constructions

Together with the lattice \mathbb{Z}^2 we often consider a graph

$$\mathbb{G} = (\mathbb{Z}^2, \mathbb{E}), \tag{4.1}$$

where \mathbb{E} is a set of edges between the neighbouring sites. Along with the discrete spaces and sets we consider “continuous” spaces (manifolds) as \mathbb{R}^2 and tori.

Now, we place the spin system on a two-dimensional torus. Let

$$\widehat{\mathbb{T}}_N = \mathbb{R}^2 / [(NL_1\mathbb{Z}) \times (NL_2\mathbb{Z})], \tag{4.2}$$

be a toric manifold. A map $\mathcal{M}_N : \mathbb{R}^2 \rightarrow \widehat{\mathbb{T}}_N$, is such that for every rectangle

$$\widehat{\Delta}_{N,n} = \{\mathbf{r} = (r_1, r_2) : L_i n N \leq r_i < L_i (n+1) N, i = 1, 2\},$$

the restricted map $\mathcal{M}_N : \widehat{\Delta}_{N,n} \rightarrow \widehat{\mathbb{T}}_N$, is a bijection. Let \mathbb{T}_N be an image of \mathbb{Z}^2 by \mathcal{M}_N

$$\mathbb{T}_N = \widehat{\mathbb{T}}_N \cap \mathcal{M}_N(\mathbb{Z}^2).$$

The coordinate system of $\widehat{\mathbb{T}}_N$ is naturally induced by \mathcal{M}_N from \mathbb{R}^2 . An ambiguity because of multivaluedness of the map \mathcal{M}_N will not lead to confusions hereinafter.

We assume that N is even. Then $\Delta_N = \widehat{\Delta}_{N,0}(0) \cap \mathbb{Z}^2$ is composed by $N^2/2$ cells of \mathbf{Z}_+ and the same amount of cells of type \mathbf{Z}_- .

Let $\Omega_N = \{-1, +1\}^{\mathbb{T}_N}$ be the set of all configurations on the torus \mathbb{T}_N . We consider Hamiltonian H_N with so called periodical boundary conditions: for any $\sigma \in \Omega_N$

$$H_N(\sigma) = -J \sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma(t)\sigma(s) - \sum_{s \in \mathbb{T}_N} h(s)\sigma(s). \quad (4.3)$$

The Gibbs measure is

$$\mu_{\beta,N}(\sigma) = \frac{1}{Z_N(\beta)} \exp\left(\beta J \sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma(t)\sigma(s) + \beta \sum_{s \in \mathbb{T}_N} h(s)\sigma(s)\right), \quad (4.4)$$

where $Z_N(\beta)$ is the corresponding partition function:

$$Z_N(\beta) = \sum_{\sigma \in \Omega_N} \exp(-\beta H_N(\sigma)). \quad (4.5)$$

4.1 Reflection Positivity and chessboard estimate

In this section we define the Reflection Positivity (RP) technique that we use. The main consequence of RP is the *chessboard estimate*, which is used to prove phase coexistence in the models with RP property. This technique was developed in the works of Fröhlich et al. [15, 16, 17, 19]. Surveys about this method can be found in Georgii [18] and Shlosman [25].

We include some detailed explanations of the RP method, because in our case there exists the dependence of chessboard estimates on the size of the cells of the external field (2.4). We will mainly use the notation and definitions of Biskup and Kotecký [5] and Biskup [4].

4.1.1 Reflection positivity.

We define reflection symmetries with respect to lines orthogonal to one of the lattice directions. Assuming the lattice \mathbb{Z}^2 embedded in \mathbb{R}^2 , we denote by Θ the group of all transformations of \mathbb{R}^2 generated by reflections of \mathbb{R}^2 with respect to lines orthogonal to one of the lattice directions such that \mathbb{Z}^2 is invariant for any $\vartheta \in \Theta$: $\vartheta\mathbb{Z}^2 = \mathbb{Z}^2$. Let ϑ_P denote the reflection ϑ with respect to the line P . The group Θ is composed by two distinct subgroups Θ^k ($k = 0, 1/2$), generated by reflections $\vartheta_{P_i^{(n,k)}}$ for which the corresponding lines are

$$P_i^{(n,k)} = \{t = (t_1, t_2) \in \mathbb{R}^2 : t_i = n + k\}, \quad (4.6)$$

for $i = 1$ or 2 , integer n and $k = 0$ or $1/2$. Reflections from Θ^0 we will call the reflections *through sites*: the corresponding reflection lines pass through the sites of \mathbb{Z}^2 . Reflections from the set $\Theta^{1/2}$ we will call reflections *through bonds*: the corresponding reflection lines bisect bonds of \mathbb{E} , (4.1).

The groups Θ^k , $k = 0, 1/2$, naturally generate the reflections of the tori $\widehat{\mathbb{T}}_N$ and \mathbb{T}_N . Thus $\vartheta_P(\widehat{\mathbb{T}}_N) = \widehat{\mathbb{T}}_N$ and $\vartheta_P(\mathbb{T}_N) = \mathbb{T}_N$. The reflecting line P in \mathbb{Z}^2 becomes two antipodal lines in the torus which splits the torus into two symmetric components, say \mathbb{T}_N^l and \mathbb{T}_N^r , the *left* and

the *right* halves. We denote those lines with the same symbol P as well as the reflection $\vartheta_P \in \Theta^k$ between the left and right halves such that $\vartheta_P(\mathbb{T}_N^l) = \mathbb{T}_N^r$ and vice versa (see Figure 1). Note that $\mathbb{T}_N^l \cap \mathbb{T}_N^r \in P$ for the reflections through the sites ($k = 0$) and are disjoint for the reflections through the bonds ($k = 1/2$).

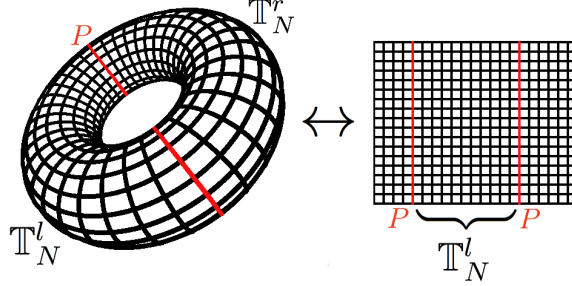


Figure 1: A torus \mathbb{T}_N is divided into the corresponding left and right half of the torus by reflecting line P passing through sites, i.e. $\vartheta_P \in \Theta^0$.

Let \mathcal{F}_P^l (\mathcal{F}_P^r) be a σ -algebra on Ω_N generated by all functions $\sigma(t), t \in \mathbb{T}_N^l$ (\mathbb{T}_N^r). As in [5] we introduce a reflection operator $\theta_P : \Omega_N \rightarrow \Omega_N$: $\theta_P(\sigma(s)) := \sigma(\vartheta_P(s))$ for any spatial reflection $\vartheta_P : \mathbb{T}_N^l \leftrightarrow \mathbb{T}_N^r$. The operator θ_P obeys the following properties:

- (1) θ_P is an involution, $\theta_P \circ \theta_P = id$;
- (2) θ_P is a *reflection* in the sense that if $\mathcal{A} \in \mathcal{F}_P^l$ depends only on configurations on $\Lambda \subset \mathbb{T}_N^l$, then $\theta_P(\mathcal{A}) \in \mathcal{F}_P^r$ depends only on configurations on $\vartheta_P(\Lambda)$.

Definition 4.1. (Reflection Positivity [15, 16], and see Definition 2.2 of [5]). *Let μ be a probability measure on Ω_N , denote E_μ the corresponding expectation, and let P be a reflecting line. We say that μ is a reflection positive measure with respect to θ_P if for any two bounded \mathcal{F}_P^l -measurable functions f and g*

$$E_\mu(f\theta_P(g)) = E_\mu(g\theta_P(f)), \quad (4.7)$$

and

$$E_\mu(f\theta_P(f)) \geq 0, \quad (4.8)$$

where $\theta_P(f)$ is the \mathcal{F}_P^r -measurable function $f \circ \theta_P$.

A consequence of RP is an inequality like the Cauchy-Schwarz inequality

$$[E_\mu(f\theta_P(g))]^2 \leq E_\mu(f\theta_P(f))E_\mu(g\theta_P(g)). \quad (4.9)$$

4.1.2 Chessboard estimates.

In this section, we recall the chessboard estimate in a form fitted to our case. The symmetries of \mathbb{T}_N which are used for the applications, are related to the symmetries of the external field. Since the external field is periodical any symmetry transformation should save block periods.

The symmetry transformation of \mathbb{T}_N are reflections of \mathbb{T}_N with respect to lines in $\widehat{\mathbb{T}}_N$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ be a set of those lines being the union of the lines

$$P_i^{(n)} = \{t = (t_1, t_2) \in \mathbb{R}^2 : t_i = nL_i + (L_i - 1)/2\}, \quad i = 1, 2, \quad (4.10)$$

where $\mathcal{P}_1 = \{P_1^{(n)}\}$ and $\mathcal{P}_2 = \{P_2^{(n)}\}$.

Note that if L_i is odd then the corresponding reflection $\vartheta_{P_i^{(n)}} \in \Theta^0$, and if L_i is even then the corresponding reflection $\vartheta_{P_i^{(n)}} \in \Theta^{1/2}$. Any such line cuts in half corresponding cells $C(n, m)$ (see (2.1)). The set of lines \mathcal{P} provides the decomposition of \mathbb{T}_N into rectangular blocks (see Figure 2(a)). In each block the total value of external field is equal to zero.

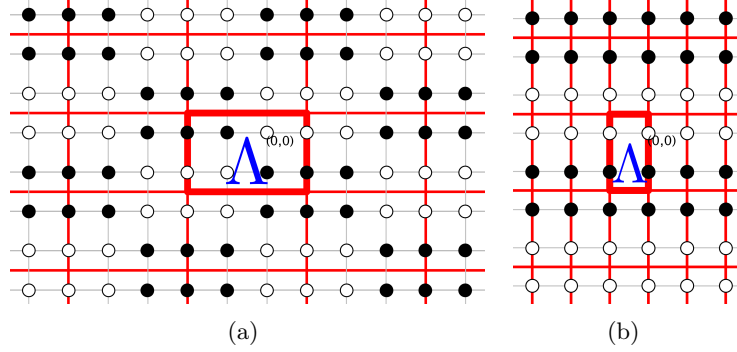


Figure 2: Representation of the external field (black and white sites) and the Λ -blocks on the torus. Red lines indicate the set \mathcal{P} of lines of reflection. (a) Cell-board model with $L_1 = 3$ and $L_2 = 2$. (b) The model studied in section 5 with $L = 2$; it can be considered as the cell-board model with $L_1 = \infty$ and $L_2 = 2$.

Let Λ be the minimal block, obtained with divisions by \mathcal{P} , which contains the origin, that is

$$\Lambda = \left\{ (t_1, t_2) \in \mathbb{T}_N : \left| t_1 + \frac{1}{2} \right| \leq \frac{L_1}{2}, \left| t_2 + \frac{1}{2} \right| \leq \frac{L_2}{2} \right\}. \quad (4.11)$$

A corresponding block $\widehat{\Lambda}$ on $\widehat{\mathbb{T}}_N$ is

$$\widehat{\Lambda} = \left\{ (r_1, r_2) \in \widehat{\mathbb{T}}_N : \left| r_1 + \frac{1}{2} \right| \leq \frac{L_1}{2}, \left| r_2 + \frac{1}{2} \right| \leq \frac{L_2}{2} \right\}. \quad (4.12)$$

Note that the block Λ contains $B_1 B_2$ sites, where

$$B_i = \begin{cases} L_i, & \text{if } L_i \text{ is even,} \\ L_i + 1, & \text{if } L_i \text{ is odd.} \end{cases} \quad (4.13)$$

The torus $\widehat{\mathbb{T}}_N$ can be covered by translations of $\widehat{\Lambda}$,

$$\widehat{\mathbb{T}}_N = \bigcup_{\mathbf{r} \in \widehat{\mathbb{T}}_N} (\widehat{\Lambda} + \mathbf{r}), \quad (4.14)$$

where $\tilde{\mathbb{T}}_N = \{\mathbf{r} = (r_1, r_2) \in \hat{\mathbb{T}}_N : r_1 = nL_1, r_2 = mL_2, n, m \in \mathbb{Z}\}$ is a quotient subgroup of \mathbb{T}_N . Correspondently the torus \mathbb{T}_N can be covered by sets $(\tilde{\Lambda} + \mathbf{r}) \cap \mathbb{T}_N$:

$$\mathbb{T}_N = \bigcup_{\mathbf{t} \in \tilde{\mathbb{T}}_N} (\Lambda + \mathbf{t}).$$

The neighboring translations of Λ can have a side in common. Let $\Omega(\Lambda) = \{\sigma_\Lambda\}$ be the set of all configurations defined on Λ and let \mathcal{F}_Λ be a σ -algebra of events on $\Omega(\Lambda)$. We call Λ -event the events $\mathcal{A} \in \mathcal{F}_\Lambda$.

Next we introduce some notions. For each $s \in \mathbb{T}_N$, the map $\tau_s : \Omega_N \rightarrow \Omega_N$ is the *translation by s* defined as $(\tau_s \sigma)(t) = \sigma(t - s)$. We consider the lines

$$Q_i = \left\{ (t_1, t_2) : t_i = -\frac{1}{2} \right\}, \quad i = 1, 2, \quad (4.15)$$

which bisect the block Λ . The reflections ϑ_{Q_1} and ϑ_{Q_2} are out of \mathcal{P} . In particular, these reflections do not shift Λ and if $\sigma \in \Omega(\Lambda)$, the corresponding operators do not preserve energy

$$H(\sigma) - H(\theta_{Q_i}(\sigma)) = -2 \sum_{t \in \Lambda} h(t) \sigma(t).$$

A *propagation* operator $\pi_{\mathbf{t}}$ on Ω_N is defined with the help of two operators

$$j_{\mathbf{t}}^{(i)} = \begin{cases} \theta_{Q_i}, & \text{if } t_i \text{ is odd,} \\ \mathbf{1}, & \text{otherwise,} \end{cases} \quad (4.16)$$

$i = 1, 2$, as following

$$\pi_{\mathbf{t}}(\sigma) = \tau_{\mathbf{t}} \circ j_{\mathbf{t}}^{(1)} \circ j_{\mathbf{t}}^{(2)}(\sigma). \quad (4.17)$$

Here $\mathbf{t} = (t_1, t_2) \in \tilde{\mathbb{T}}_N$. The symbol $\mathbf{1}$ above means the identical operator $\mathbf{1}(\sigma) = \sigma$. The configuration σ is shifted such that its values on Λ is moved to $\Lambda + \mathbf{t}$, with the possible reflections θ_{Q_1} and θ_{Q_2} , depending on the parities of t_1 and t_2 . The event $\pi_{\mathbf{t}}(\mathcal{A})$ is a cylindrical set of configurations from $\mathcal{F}_{\Lambda + \mathbf{t}}$.

We remark that a propagator is based on reflection through the sides between two neighbors blocks. Let $\Pi_{\tilde{\mathbb{T}}_N}$ mean the set of all propagations corresponding the torus $\tilde{\mathbb{T}}_N$. Any propagation $\pi_{\mathbf{t}} \in \Pi_{\tilde{\mathbb{T}}_N}$ is a bijection

$$\sigma \in \Omega(\Lambda) \leftrightarrow \pi_{\mathbf{t}}(\sigma) \in \Omega(\Lambda + \mathbf{t}).$$

Thus we can use the inverse map $\pi_{\mathbf{t}}^{-1}$.

As we will see in the proof of Proposition 4.5, depending on parity of L_1 and L_2 , we work with four similar propagators $\pi^{*(k)}$, $k \in \{(h, 1), (h, 2), (v, 1), (v, 2)\}$, related to *double-blocks* or Λ^* -blocks. Particularly, when L_1 (resp. L_2) is even, we work with horizontal (resp. vertical) Λ^* -blocks, defined by

$$\begin{aligned} \Lambda_{h,1}^* &= \Lambda \cup (\Lambda + (L_1, 0)), & \Lambda_{h,2}^* &= \Lambda \cup (\Lambda - (L_1, 0)), \\ \Lambda_{v,1}^* &= \Lambda \cup (\Lambda + (0, L_2)), & \Lambda_{v,2}^* &= \Lambda \cup (\Lambda - (0, L_2)). \end{aligned} \quad (4.18)$$

The associated quotient subgroups of \mathbb{T}_N are, respectively, given by

$$\begin{aligned}
\tilde{\mathbb{T}}_N^{(h,1)} &= \{\mathbf{t} = (t_1, t_2) \in \tilde{\mathbb{T}}_N : t_1 = 2nL_1, t_2 = mL_2\}, \\
\tilde{\mathbb{T}}_N^{(h,2)} &= \{\mathbf{t} = (t_1, t_2) \in \tilde{\mathbb{T}}_N : t_1 = (2n-1)L_1, t_2 = mL_2\}, \\
\tilde{\mathbb{T}}_N^{(v,1)} &= \{\mathbf{t} = (t_1, t_2) \in \tilde{\mathbb{T}}_N : t_1 = nL_1, t_2 = 2mL_2\}, \\
\tilde{\mathbb{T}}_N^{(v,2)} &= \{\mathbf{t} = (t_1, t_2) \in \tilde{\mathbb{T}}_N : t_1 = nL_1, t_2 = (2m-1)L_2\}.
\end{aligned} \tag{4.19}$$

Further we use a notion Λ^* -events for sets configurations defined on each of the Λ_k^* -blocks in (4.18). Moreover, for brevity we denote

$$\mathcal{D} := \{(h, 1), (h, 2), (v, 1), (v, 2)\}. \tag{4.20}$$

Now, we state the chessboard estimates.

Theorem 4.2. (Chessboard estimate [15, 16, 4, 25]) *Let $\mu_{\beta, N}$ a measure on Ω_N which is RP with respect to all reflections between the neighboring blocks $\Lambda + \mathbf{t}$, $\mathbf{t} \in \tilde{\mathbb{T}}_N$. Then for any Λ -events $\mathcal{A}_1, \dots, \mathcal{A}_m$ and any distinct sites $\mathbf{t}_1, \dots, \mathbf{t}_m \in \tilde{\mathbb{T}}_N$,*

$$\mu_{\beta, N} \left(\bigcap_{j=1}^m \pi_{\mathbf{t}_j}(\mathcal{A}_j) \right) \leq \prod_{j=1}^m \mu_{\beta, N} \left(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{A}_j) \right)^{1/|\tilde{\mathbb{T}}_N|}. \tag{4.21}$$

Moreover, for any Λ^* -events $\mathcal{A}_1, \dots, \mathcal{A}_m$ and any distinct sites $\mathbf{t}_1, \dots, \mathbf{t}_m \in \tilde{\mathbb{T}}_N^{(k)}$, $k \in \mathcal{D}$,

$$\mu_{\beta, N} \left(\bigcap_{j=1}^m \pi_{\mathbf{t}_j}^{*(k)}(\mathcal{A}_j) \right) \leq \prod_{j=1}^m \mu_{\beta, N} \left(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}^{*(k)}(\mathcal{A}_j) \right)^{1/|\tilde{\mathbb{T}}_N^{(k)}|}. \tag{4.22}$$

For the proof see, for example, [4], Theorem 5.8 or [5], Theorem 2.4.

The following quantities play the main role in the proof of the phase transition

$$\mathfrak{z}_{\beta, N}(\mathcal{A}) := \mu_{\beta, N} \left(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{A}) \right)^{1/|\tilde{\mathbb{T}}_N|}, \tag{4.23}$$

where \mathcal{A} is Λ -event. The function $\mathcal{A} \rightarrow \mathfrak{z}_{\beta, N}(\mathcal{A})$ is not additive. However, given the σ -additivity of μ and using the chessboard estimate, it is easy to prove that it is sub-additive (see [4], Lemma 5.9). That is, for any collection of Λ -events $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \dots$ such that $\mathcal{A} \subset \cup_l \mathcal{A}_l$, the inequality

$$\mathfrak{z}_{\beta, N}(\mathcal{A}) \leq \sum_l \mathfrak{z}_{\beta, N}(\mathcal{A}_l) \tag{4.24}$$

holds. The limiting version of this quantity will be of particular interest for us. Thus, we define

$$\mathfrak{z}_{\beta}(\mathcal{A}) := \lim_{N \rightarrow \infty} \mathfrak{z}_{\beta, N}(\mathcal{A}). \tag{4.25}$$

The existence of the limit follows from the sub-additivity. Furthermore, we define for Λ_k^* -event \mathcal{A} , where $k \in \mathcal{D}$, the similar quantities

$$\mathfrak{z}_{\beta, N}^{(k)} := \mu_{\beta, N} \left(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}^{*(k)}(\mathcal{A}) \right)^{1/|\tilde{\mathbb{T}}_N^{(k)}|}, \tag{4.26}$$

and

$$\mathfrak{z}_{\beta}^{(k)}(\mathcal{A}) := \lim_{N \rightarrow \infty} \mathfrak{z}_{\beta, N}^{(k)}(\mathcal{A}).$$

4.2 Phase coexistence

The basis of the proof of Theorem 2.2 are Propositions 4.3, 4.4 and 4.5 which we shall prove in the subsection 4.3.

The main applied technics is the reflection positivity technique. The proof essentially consists on two steps. First, the easiest step, Proposition 4.3, we apply a known criterion for establish RP property for our model. Second, we construct two measures μ_β^+ and μ_β^- and prove that the probabilities $\mu_\beta^+(\sigma(0) = -1)$ and $\mu_\beta^-(\sigma(0) = +1)$ can be made less than $1/2$ for large β . This will prove the phase coexistence. In order to provide this we use the chessboard estimate (4.21) for a sort of Peierls arguments evaluating the contour probabilities. It is implemented in the proof of Proposition 4.5.

The contour technique we work with are based on usage of thick contours partly assembled of a block set $\{\Lambda + \mathbf{t}\}$, where $\mathbf{t} \in \tilde{\mathbb{T}}_N$ and partly assembled of the double-blocks $\{\Lambda^*\}$. We shall describe later all details.

We need some preliminary results about $\mathfrak{z}_{\beta,N}$ and $\mathfrak{z}_{\beta,N}^{(k)}$, $k \in \mathcal{D}$ (see (4.20)), defined in (4.23) and (4.26).

Proposition 4.3. *For any $P \in \mathcal{P}$ (see (4.10) and below), and all $\beta \geq 0$ the Gibbs measure $\mu_{\beta,N}$ (4.4) on the torus \mathbb{T}_N is reflection positive (RP) with respect to θ_P .*

In order to apply the chessboard inequality we introduce *bad block events* we deal with. Let σ_Λ^+ and σ_Λ^- be the constant configurations on Λ with all spins plus and all spins minus, respectively.

For each configuration $\sigma_\Lambda \in \{-1, +1\}^\Lambda$ on Λ we define the event

$$\mathcal{B}(\sigma_\Lambda) = \{\sigma \in \Omega_N : \sigma(\Lambda) = \sigma_\Lambda\}. \quad (4.27)$$

Let $R(\Lambda)$ be the set of all Λ -bad configurations, $R(\Lambda) = \{-1, +1\}^\Lambda \setminus \{\sigma_\Lambda^+, \sigma_\Lambda^-\}$. Remember that the size of the block Λ is equal to $B_1 B_2$ sites, and $B_i \geq 2$ as defined in (4.13). This implies that $|R(\Lambda)| = 2^{B_1 B_2} - 2 \geq 14$.

Let \mathcal{R}_Λ denote the event that the block Λ is σ -bad for $\sigma \in \Omega_N$, that is,

$$\mathcal{R}_\Lambda = \{\sigma \in \Omega_N : \sigma(\Lambda) \neq \sigma_\Lambda^\pm\} = \bigcup_{\sigma_\Lambda \in R(\Lambda)} \mathcal{B}(\sigma_\Lambda). \quad (4.28)$$

The event \mathcal{R}_Λ is called Λ -bad event. Which represents all the torus configurations that are not constant on Λ -block.

The proof of the main theorem about the phase coexistence is based on the contour technique. Both Peierls contours and thick contours are applied. The thick contours are consisted of the Λ -blocks and Λ^* -blocks. The Λ -block is included to the thick contour if Peierls contour touches this Λ -block, the Λ^* -block appears in the thick contour when Peierls contour at least partly passes between neighbouring Λ -blocks. It happens when the size L_i in the direction of the block localisations is even.

If $\mathbf{t} \in \{(L_1, 0), (0, L_2)\}$ then the neighbouring Λ -blocks Λ and $\Lambda + \mathbf{t}$ are not intersected, that is $\Lambda \cap (\Lambda + (L_1, 0)) = \emptyset$ ($\Lambda \cap (\Lambda + (0, L_2)) = \emptyset$) when L_1 (L_2) is even.

As in (4.28) we define the Λ^* -bad-block events,

$$\mathcal{R}_{\Lambda_k^*} = \{\sigma \in \Omega_N : \sigma(\Lambda_k^*) \neq \sigma_{\Lambda_k^*}^\pm\}, \quad (4.29)$$

where $\sigma_{\Lambda^*}^+$ and $\sigma_{\Lambda^*}^-$ are the constant configurations on each of the Λ_k^* -blocks, $k \in \mathcal{D}$.

In the next proposition we show that the Λ -bad and Λ^* -bad events have small probability independently on N , when β is large.

Proposition 4.4. *If the condition (2.8) holds true, then for any even N*

$$\mathfrak{z}_{\beta,N}(\mathcal{R}_\Lambda) \leq 2^{B_1 B_2} \exp \left\{ -\beta \left(2J - \frac{hL_1 L_2}{L_1 + L_2} \right) \right\}. \quad (4.30)$$

Moreover, for any $k \in \mathcal{D}$, and any N multiple of 4,

$$\mathfrak{z}_{\beta,N}^{(k)}(\mathcal{R}_{\Lambda_k^*}) \leq 4^{B_1 B_2} \exp \left\{ -\beta \left(2J - \frac{hL_1 L_2}{L_1 + L_2} \right) \right\}. \quad (4.31)$$

Now, we can state the main proposition.

Proposition 4.5. *Let the condition (2.8) holds true. There exists a constant $c > 1$ such that for any $s, t \in \mathbb{T}_N$, the following inequality holds*

$$\mu_{\beta,N}(\sigma(s) = +1, \sigma(t) = -1) \leq 2c(c+1)2^{B_1 B_2/2} \exp \left\{ -\frac{\beta}{8} \left(2J - \frac{hL_1 L_2}{L_1 + L_2} \right) \right\}, \quad (4.32)$$

for any

$$\beta > \beta' = \frac{4[(B_1 B_2 + 2) \ln 2 + 2 \ln(c(c+1))]}{2J - \frac{hL_1 L_2}{L_1 + L_2}}. \quad (4.33)$$

The constant c appears from the combinatorial argument related to the number of contours builded from n Λ - and Λ^* -blocks.

4.2.1 Proof of Theorem 2.2.

First of all, we use the following symmetry of the torus measure. Let Λ_s be the block containing the site $s \in \mathbb{T}_N$, then

$$\mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv +1) = \mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv -1) = \frac{1 - \mu_{\beta,N}(\mathcal{R}(\Lambda_s))}{2}, \quad (4.34)$$

for any $s \in \mathbb{T}_N$. In order to check this equality we apply the following two transformations for any configuration $\sigma \in \Omega_N$, such that $\sigma(\Lambda) \equiv +1$. First, we apply on σ the reflection operator θ_{Q_1} , defined as in section 4.1, where Q_1 is given by (4.15). That is, $\omega := \theta_{Q_1}(\sigma) \in \Omega_N$ takes the value $\omega(t_1, t_2) = \sigma(-t_1 - 1, t_2)$, for all $t = (t_1, t_2) \in \mathbb{T}_N$. Second, we obtain $\sigma' = -\omega$, flipping all the spin values. In other words, $\sigma'(t) = -\sigma(-t_1 - 1, t_2)$, for all $t \in \mathbb{T}_N$. Clearly, $\sigma'(\Lambda) \equiv -1$ and given $h_{(t_1, t_2)} = -h_{(-t_1 - 1, t_2)}$, the Hamiltonians are equal $H_N(\sigma) = H_N(\sigma')$. It proves (4.34).

Since the symmetry property of the model the following equalities hold

$$\mu_{\beta,N}(\mathcal{R}_\Lambda) = \mu_{\beta,N}(\mathcal{R}_{\Lambda+\mathbf{t}}),$$

for any $\mathbf{t} \in \widetilde{\mathbb{T}}_N$. Therefore we omit sometimes the index Λ at \mathcal{R} .

Using the chessboard estimate (4.21), we obtain the inequality

$$\mu_{\beta,N}(\mathcal{R}) \leq \mu_{\beta,N} \left(\bigcap_{\mathbf{t} \in \widetilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{R}) \right)^{1/|\widetilde{\mathbb{T}}_N|} = \mathfrak{z}_{\beta,N}(\mathcal{R}). \quad (4.35)$$

Then, from (4.34)

$$\mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv +1) \geq \frac{1 - \mathfrak{z}_{\beta,N}(\mathcal{R})}{2}. \quad (4.36)$$

Let $t \in \mathbb{T}_N$ such that $t = s + (NL_1/2, 0)$, and define

$$\mu_{\beta,N}^{\pm}(\cdot) := \mu_{\beta,N}(\cdot \mid \sigma(t) = \pm 1). \quad (4.37)$$

By (4.36), (4.30), and Proposition 4.5 we have

$$\begin{aligned} \mu_{\beta,N}^+(\sigma(s) = -1) &\leq \frac{\mu_{\beta,N}(\sigma(s) = -1, \sigma(t) = +1)}{\mu_{\beta,N}(\sigma : \sigma(\Lambda_t) \equiv +1)} \\ &\leq \frac{4c(c+1)2^{B_1B_2/2} \exp\left\{-\frac{\beta}{8}\left(2J - h\frac{L_1L_2}{L_1+L_2}\right)\right\}}{1 - 2^{B_1B_2} \exp\left\{-\beta\left(2J - h\frac{L_1L_2}{L_1+L_2}\right)\right\}}, \end{aligned} \quad (4.38)$$

and

$$\mu_{\beta,N}^-(\sigma(s) = +1) \leq \frac{4c(c+1)2^{B_1B_2/2} \exp\left\{-\frac{\beta}{8}\left(2J - h\frac{L_1L_2}{L_1+L_2}\right)\right\}}{1 - 2^{B_1B_2} \exp\left\{-\beta\left(2J - h\frac{L_1L_2}{L_1+L_2}\right)\right\}}. \quad (4.39)$$

When $N \nearrow \infty$ we extract from the sequences of the measures $(\mu_{\beta,N}^+)$ and $(\mu_{\beta,N}^-)$, two converging subsequences. Let μ_{β}^+ and μ_{β}^- be corresponding limits. Those measures are infinite-volume Gibbs measures corresponding to the Hamiltonian H ((2.3) and (2.4)). It follows from DLR-equation that those measures are Gibbsian (see [4]).

By (4.38) and (4.39) the inequalities (2.11) are satisfied if

$$16c(c+1)2^{B_1B_2} \exp\left\{-\frac{\beta}{8}\left(2J - h\frac{L_1L_2}{L_1+L_2}\right)\right\} < 1. \quad (4.40)$$

This inequality means that the phase transition holds for all

$$\beta > \frac{8[(B_1B_2 + 4)\ln 2 + \ln(c(c+1))]}{2J - \frac{hL_1L_2}{L_1+L_2}}. \quad (4.41)$$

□

4.3 Remaining proofs

4.3.1 Proof of Proposition 4.3.

The proof is the application of the known criteria for a measure to be reflection positive. Fix a line $P \in \mathcal{P}$ of the reflections and let θ_P be the corresponding reflection operator. The criteria applied to our case claims that the measure $\mu_{\beta,N}$ is reflection positive, if its Hamiltonian can be represented in the form

$$-H_N = A + \theta_P(A) + \sum_{\alpha} C_{\alpha} \theta_P(C_{\alpha}), \quad (4.42)$$

where A, C_{α} are \mathcal{F}_P^l -measurable functions. Then for all $\beta \geq 0$ the torus Gibbs measure $\mu_{\beta,N}$, is RP with respect to θ_P (see Definition 4.1). The criteria can be found in Theorem 2.1 of Shlosman [25] or Corollary 5.4 of Biskup [4].

In our case there are two possibilities for $P \in \mathcal{P}$: P passes through sites of \mathbb{T}_N or not. In the case of P passing through the sites of \mathbb{T}_N choose

$$A = J \sum_{\substack{\langle t,s \rangle: \\ t \in \mathbb{T}_N^l, s \in \mathbb{T}_N^l \setminus P}} \sigma(t)\sigma(s) + \frac{J}{2} \sum_{\langle t,s \rangle \in P} \sigma(t)\sigma(s) + \sum_{s \in \mathbb{T}_N^l \setminus P} h(s)\sigma(s) + \frac{1}{2} \sum_{s \in P} h(s)\sigma(s),$$

then, since $h(s) = h(\vartheta_P(s))$,

$$-H_N(\sigma) = A + \theta_P(A),$$

here functions $C_\alpha \equiv 0$. In the case of reflections through the bonds choose

$$A = J \sum_{\langle t,s \rangle \in \mathbb{T}_N^l} \sigma(t)\sigma(s) + \sum_{s \in \mathbb{T}_N^l} h(s)\sigma(s),$$

then

$$-H_N(\sigma) = A + \theta_P(A) + J \sum_{\substack{t \in \mathbb{T}_N^l: \\ |t-P|=1/2}} \sigma(t)\theta_P(\sigma(t)).$$

The equality $h(s) = h(\vartheta_P(s))$ is used again. That proves the proposition. \square

4.3.2 Proof of Proposition 4.4.

For simplicity, we only prove (4.30), but the proof for each Λ_k^* -bad-event, $k \in \mathcal{D}$ is the same. We justify the condition N multiple of 4 in (4.31), since the number of Λ -blocks fulfilling the whole torus \mathbb{T}_N , is twice the number of Λ^* -blocks needed.

Let $\sigma_{\Lambda,N} := \cap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{B}(\sigma_\Lambda))$ be the configuration on \mathbb{T}_N , obtained by the propagations of a fixed block configuration σ_Λ . The proof of the proposition 4.4 will be based on the following inequality

$$\mathfrak{z}_{\beta,N}(\mathcal{B}(\sigma_\Lambda))^{|\tilde{\mathbb{T}}_N|} = \frac{\exp(-\beta H_N(\sigma_{\Lambda,N}))}{Z_N(\beta)} \leq \exp(-\beta [H_N(\sigma_{\Lambda,N}) - H_N(\sigma^+)]). \quad (4.43)$$

A bound of the right hand side of (4.43) can be found from the next two lemmas.

In order to formulate the first lemma we introduce some notions. Consider the configuration $\sigma_{\Lambda,N}$ defined above. For any σ_Λ the configuration $\sigma_{\Lambda,N}$ has the following periodicity property: for any $t \in \mathbb{T}_N$ and $\mathbf{t} \in \tilde{\mathbb{T}}_N$ we have

$$\sigma_{\Lambda,N}(t) = \sigma_{\Lambda,N}(t + 2\mathbf{t}). \quad (4.44)$$

It means that there exists some “minimal” sublattice $\Lambda^{[2 \times 2]}$ of \mathbb{T}_N , such that the configuration $\sigma_{\Lambda,N}$ can be obtained by translations of $\sigma_{\Lambda,N}(\Lambda^{[2 \times 2]})$. Indeed, using the rectangle $\hat{\Lambda}$ defined by (4.12) let us define

$$\begin{aligned} \hat{\Lambda}^{[2 \times 2]} &:= \hat{\Lambda} \cup \left(\hat{\Lambda} + (0, L_2) \right) \cup \left(\hat{\Lambda} + (L_1, 0) \right) \cup \left(\hat{\Lambda} + (L_1, L_2) \right) + \left(\frac{1}{4}, \frac{1}{4} \right), \\ \Lambda^{[2 \times 2]} &:= \mathbb{T}_N \cap \hat{\Lambda}^{[2 \times 2]}. \end{aligned} \quad (4.45)$$

See Figure 3 for illustration.

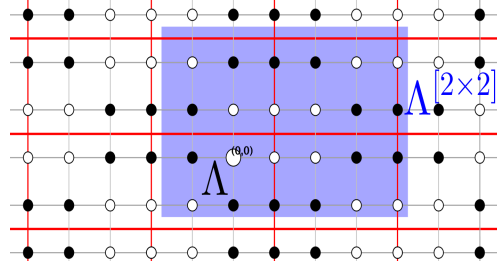


Figure 3: The sublattice $\Lambda^{[2 \times 2]}$ is composed by the sites inside the blue rectangle $\widehat{\Lambda}^{[2 \times 2]}$. Here $L_1 = 3$ and $L_2 = 2$.

Remark that $(\Lambda^{[2 \times 2]} + 2\mathbf{t}_1) \cap (\Lambda^{[2 \times 2]} + 2\mathbf{t}_2) = \emptyset$, if $\mathbf{t}_1 \neq \mathbf{t}_2$, with $\mathbf{t}_1, \mathbf{t}_2 \in \widetilde{\mathbb{T}}_N$. For any $\mathbf{t} \in \widetilde{\mathbb{T}}_N$, we have $h(s) = h(s + 2\mathbf{t})$, as well as,

$$\sigma_{\Lambda, [2 \times 2]} := \sigma_{\Lambda, N}(\Lambda^{[2 \times 2]}) = \sigma_{\Lambda, N}(\Lambda^{[2 \times 2]} + 2\mathbf{t}) \text{ and } \mathbb{T}_N = \bigcup_{\mathbf{t} \in \widetilde{\mathbb{T}}_N} (\Lambda^{[2 \times 2]} + 2\mathbf{t}). \quad (4.46)$$

Let $\widetilde{\mathbb{T}}_N^{[2 \times 2]} = \{\mathbf{t} = (t_1, t_2) \in \mathbb{T}_N : t_1 = 2nL_1, t_2 = 2mL_2, n, m \in \mathbb{Z}\}$ and $\mathbb{T}^{[2 \times 2]} = \mathbb{T}_N / \widetilde{\mathbb{T}}_N^{[2 \times 2]}$. Let us define the torus Hamiltonian: for any $\sigma \in \{-1, +1\}^{\mathbb{T}^{[2 \times 2]}}$

$$H_{[2 \times 2]}(\sigma) = -J \sum_{\langle t, s \rangle \in \mathbb{T}^{[2 \times 2]}} \sigma(t)\sigma(s) - \sum_{s \in \mathbb{T}^{[2 \times 2]}} h(s)\sigma(s). \quad (4.47)$$

The sites $(-\lfloor \frac{2L_1+1}{4} \rfloor, t_2)^1$ and $(\lfloor \frac{3L_1-1}{2} \rfloor, t_2) \in \Lambda^{[2 \times 2]}$, are neighbours in $\mathbb{T}^{[2 \times 2]}$ for any $t_2 \in [-\lfloor \frac{2L_2+1}{4} \rfloor, \lfloor \frac{3L_2-1}{2} \rfloor]$. As well as, $(t_1, -\lfloor \frac{2L_2+1}{4} \rfloor)$ and $(t_1, \lfloor \frac{3L_2-1}{2} \rfloor) \in \Lambda^{[2 \times 2]}$, are neighbours when $t_1 \in [-\lfloor \frac{2L_1+1}{4} \rfloor, \lfloor \frac{3L_1-1}{2} \rfloor]$.

Lemma 4.6. *For any configuration $\sigma_\Lambda \in \{-1, +1\}^\Lambda$, then*

$$H_N(\sigma_{\Lambda, N}) = \left(\frac{N}{2}\right)^2 H_{[2 \times 2]}(\sigma_{\Lambda, [2 \times 2]}). \quad (4.48)$$

Proof. Note that the number of sites in $\widetilde{\mathbb{T}}_N^{[2 \times 2]}$ is equal to $(N/2)^2$. Therefore, for the external field part in the Hamiltonian we have

$$\begin{aligned} \sum_{s \in \mathbb{T}_N} h(s)\sigma_{\Lambda, N}(s) &= \sum_{\mathbf{t} \in \widetilde{\mathbb{T}}_N^{[2 \times 2]}} \left(\sum_{s \in \Lambda^{[2 \times 2]} + \mathbf{t}} h(s)\sigma_{\Lambda, N}(s) \right) \\ &= \left(\frac{N}{2}\right)^2 \sum_{s \in \mathbb{T}^{[2 \times 2]}} h(s)\sigma_{\Lambda, [2 \times 2]}(s). \end{aligned} \quad (4.49)$$

For the interaction part we can write

¹ $\lfloor x \rfloor$, denotes the floor function, that is: $\lfloor x \rfloor = \max \{m \in \mathbb{Z} \mid m \leq x\}$.

$$\sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma(t)\sigma(s) = \sum_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{[2 \times 2]}} \left(\sum_{\langle t,s \rangle \in \Lambda^{[2 \times 2] + \mathbf{t}}} \sigma(t)\sigma(s) + \sum_{\substack{\langle t,s \rangle: t \in (\Lambda^{[2 \times 2] + \mathbf{t}}), \\ s \in (\Lambda^{[2 \times 2] + \mathbf{t} + (2L_1, 0)}}} \sigma(t)\sigma(s) + \sum_{\substack{\langle t,s \rangle: t \in (\Lambda^{[2 \times 2] + \mathbf{t}}), \\ s \in (\Lambda^{[2 \times 2] + \mathbf{t} + (0, 2L_2)}}} \sigma(t)\sigma(s) \right), \quad (4.50)$$

for any $\sigma \in \Omega_N$. We apply this representation for configuration $\sigma_{\Lambda, N}$, and remembering the periodicity condition (4.46) we obtain

$$\begin{aligned} \sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma_{\Lambda, N}(t)\sigma_{\Lambda, N}(s) &= \sum_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{[2 \times 2]}} \sum_{\langle t,s \rangle \in \mathbb{T}^{[2 \times 2]}} \sigma_{\Lambda, [2 \times 2]}(t)\sigma_{\Lambda, [2 \times 2]}(s) \\ &= \left(\frac{N}{2}\right)^2 \sum_{\langle t,s \rangle \in \mathbb{T}^{[2 \times 2]}} \sigma_{\Lambda, [2 \times 2]}(t)\sigma_{\Lambda, [2 \times 2]}(s). \end{aligned} \quad (4.51)$$

The relation (4.48) follows from (4.49) and (4.51). \square

Since, the number of sites in $\tilde{\mathbb{T}}_N$ is equal to N^2 , using Lemma 4.6 we obtain from (4.43)

$$\mathfrak{z}_{\beta, N}(\mathcal{B}(\sigma_{\Lambda})) \leq \exp\left(-\frac{\beta}{4} \left[H_{[2 \times 2]}(\sigma_{\Lambda, [2 \times 2]}) - H_{[2 \times 2]}(\sigma_{\Lambda, [2 \times 2]}^+) \right]\right), \quad (4.52)$$

for any $\sigma_{\Lambda} \in R(\Lambda)$, where $\sigma_{\Lambda, [2 \times 2]}^+$ is +1 constant configuration on $\mathbb{T}^{[2 \times 2]}$.

Finally the Proposition 4.4 readily follows from the next lemma. The proof of this lemma is essentially repeat the arguments of Lemma 3.3 from [22], but we provide the proof for completeness.

Lemma 4.7. *If h satisfies (2.8) then for all configuration $\sigma_{\Lambda} \in R(\Lambda)$, its extended configuration $\sigma_{\Lambda, [2 \times 2]}$, defined by (4.46), on torus $\mathbb{T}^{[2 \times 2]}$ satisfies*

$$H_{[2 \times 2]}(\sigma_{\Lambda, [2 \times 2]}) - H_{[2 \times 2]}(\sigma_{\Lambda, [2 \times 2]}^+) \geq 4 \left(2J - h \frac{L_1 L_2}{L_1 + L_2} \right). \quad (4.53)$$

Proof. The proof is taken from [22]. For brevity of notations, proving (4.53) we omit the indices $\Lambda, [2 \times 2]$ everywhere for the configurations from $\mathbb{T}^{[2 \times 2]}$. Let for $V \subset \mathbb{T}^{[2 \times 2]}$ the configuration σ^V be the perturbation of σ , that is

$$\sigma^V(t) = \begin{cases} -\sigma(t), & \text{if } t \in V, \\ \sigma(t), & \text{if } t \notin V. \end{cases} \quad (4.54)$$

Let denote $\sigma^{V,+}$ the perturbation of σ^+ .

We prove the inequality (4.53) for all $V \neq \emptyset$ and $V \neq \mathbb{T}^{[2 \times 2]}$. Since $V \subset \mathbb{T}^{[2 \times 2]}$ there are the edges $\langle u, v \rangle \subset \mathbb{T}^{[2 \times 2]}$ such that $u \in V, v \in V^c$, thus $\sigma^{V,+}(u) = -1, \sigma^{V,+}(v) = +1$. The set of those edges composes Peierls contour (or simply, a contour) ∂V of V . Peierls contour is union of the horizontal edges $\partial^h V$ and the vertical edges $\partial^v V$, i.e. $\partial V = \partial^h V \cup \partial^v V$. Note that,

$$H_{[2 \times 2]}(\sigma^{V,+}) - H_{[2 \times 2]}(\sigma^+) = 2J|\partial V| + 2 \sum_{s \in V} h(s). \quad (4.55)$$

We represent V as union of sets $\mathcal{S}_V = \{S\}$ of connected horizontal lines of the sites or as union of sets $\mathcal{T}_V = \{T\}$ of connected vertical lines of the sites. Then

$$\sum_{s \in V} h(s) = \sum_{S \in \mathcal{S}_V} \sum_{s \in S} h(s) = \sum_{T \in \mathcal{T}_V} \sum_{s \in T} h(s). \quad (4.56)$$

There are two types of the connected lines of the sites in V : *closed* or *open*. In the last case the line has two ends which belong to the boundary edges of ∂V . If the open line is horizontal then there exist two edges of $\partial^h V$ having common points with the line. The same property is true for the vertical open lines intersecting $\partial^v V$. The closed lines do not intersect the boundary ∂V . Their are closed circles around the torus $\mathbb{T}^{[2 \times 2]}$.

Let $\mathcal{S}_V = \mathcal{S}_V^{cl} \cup \mathcal{S}_V^{op}$, where \mathcal{S}_V^{cl} is the subset of the closed horizontal lines and \mathcal{S}_V^{op} is the subset of the open horizontal lines. The similar representation holds for the vertical lines, $\mathcal{T}_V = \mathcal{T}_V^{cl} \cup \mathcal{T}_V^{op}$. For the closed lines

$$\sum_{s \in S} h(s) = \sum_{s \in T} h(s) = 0, \quad (4.57)$$

where $S \in \mathcal{S}_V^{cl}$ and $T \in \mathcal{T}_V^{cl}$. For the open lines the following estimates hold

$$\begin{aligned} \left| \sum_{s \in S} h(s) \right| &= h |S \cap \mathbf{Z}_+| - |S \cap \mathbf{Z}_-| \leq h L_1, \\ \left| \sum_{s \in T} h(s) \right| &= h |T \cap \mathbf{Z}_+| - |T \cap \mathbf{Z}_-| \leq h L_2, \end{aligned} \quad (4.58)$$

where $S \in \mathcal{S}_V^{op}$ and $T \in \mathcal{T}_V^{op}$. This implies the following inequalities

$$\begin{aligned} 2 \sum_{S \in \mathcal{S}_V} \sum_{s \in S} h(s) &\geq -2h L_1 |\mathcal{S}_V| \geq -h L_1 |\partial^h V|, \\ 2 \sum_{T \in \mathcal{T}_V} \sum_{s \in T} h(s) &\geq -2h L_2 |\mathcal{T}_V| \geq -h L_2 |\partial^v V|. \end{aligned} \quad (4.59)$$

Finally, by (4.55), (4.56) and (4.59),

$$\begin{aligned} H_{[2 \times 2]}(\sigma^{V,+}) - H_{[2 \times 2]}(\sigma^+) &= 2J |\partial V| + \frac{L_1}{L_1 + L_2} \left(2 \sum_{s \in V} h(s) \right) + \frac{L_2}{L_1 + L_2} \left(2 \sum_{s \in V} h(s) \right) \\ &\geq 2J |\partial V| - h \frac{L_1 L_2}{L_1 + L_2} |\partial^h V| - h \frac{L_1 L_2}{L_1 + L_2} |\partial^v V| \\ &= \left(2J - h \frac{L_1 L_2}{L_1 + L_2} \right) |\partial V|. \end{aligned}$$

Clearly for each $V \subset \mathbb{T}^{[2]}$, $|\partial V| \geq 4$. We have proved (4.53). \square

Finally, applying this lemma to (4.52), and by sub-additivity of $\mathfrak{z}_{\beta,N}$ (see (4.24)), we have

$$\begin{aligned} \mathfrak{z}_{\beta,N}(\mathcal{R}) &\leq \sum_{\sigma_\Lambda \in R(\Lambda)} \mathfrak{z}_{\beta,N}(\mathcal{B}(\sigma_\Lambda)) \\ &\leq \sum_{\sigma_\Lambda \in R(\Lambda)} \exp \left(-\beta \left(2J - h \frac{L_1 L_2}{L_1 + L_2} \right) \right) \\ &= |R(\Lambda)| \exp \left(-\beta \left(2J - h \frac{L_1 L_2}{L_1 + L_2} \right) \right). \end{aligned} \quad (4.60)$$

The number $|R(\Lambda)|$ of the bad configurations on Λ is estimated as $|R(\Lambda)| \leq 2^{B_1 B_2}$. It proves (4.30).

The inequality (4.31) we obtain by the same way as (4.30) was proved, using the estimate $|R(\Lambda^*)| \leq 4^{B_1 B_2}$. \square

4.3.3 Proof of Proposition 4.5.

Denote $\Omega_N^{s,t} = \{\sigma \in \Omega_N : \sigma(s) = +1 \text{ and } \sigma(t) = -1\}$. Then for each $\sigma \in \Omega_N^{s,t}$ we define the set

$$I^+(\sigma) = \{u \in \mathbb{T}_N : \sigma(u) = +1\}, \quad (4.61)$$

and let $I^+(\sigma, s) \subseteq I^+(\sigma)$ be its maximal connected component containing the site s . The sites s and t are separated by some Peierls contour $\gamma(\sigma)$. As above Peierls contour is the set of edges $\{\langle u, v \rangle\}$ such that $\sigma(u) \neq \sigma(v)$.

We define a dual to $\gamma(\sigma)$ contour $\gamma^*(\sigma)$. To this end we consider the dual lattice \mathbb{Z}^{*2} and a dual graph $\mathbb{G}^* = (\mathbb{Z}^{*2}, \mathbb{E}^*)$. The edges $\langle u^*, v^* \rangle \in \mathbb{E}^*$ are orthogonal to the edges from \mathbb{E} (see (4.1)). The dual to $\gamma(\sigma)$ contour $\gamma^*(\sigma) = \langle u^*, v^* \rangle$ consists of the all dual edges which are orthogonal to the edges from $\gamma(\sigma)$.

More formal description is as following. An edge $\langle u^*, v^* \rangle$ with $u^* = (u_1^*, u_2^*)$ and $v^* = (v_1^*, v_2^*)$ dual to $\langle u, v \rangle$ with $u = (u_1, u_2)$ and $v = (v_1, v_2)$, is defined by

$$\begin{aligned} u_1^* &= \frac{u_1+v_1}{2}|u_1-v_1| + (u_1 + \frac{1}{2})|u_2-v_2|, \\ u_2^* &= \frac{u_2+v_2}{2}|u_2-v_2| + (u_2 + \frac{1}{2})|u_1-v_1|, \\ v_1^* &= \frac{u_1+v_1}{2}|u_1-v_1| + (v_1 - \frac{1}{2})|u_2-v_2|, \\ v_2^* &= \frac{u_2+v_2}{2}|u_2-v_2| + (v_2 - \frac{1}{2})|u_1-v_1|. \end{aligned}$$

Let $\gamma^{ext}(\sigma, s) \subseteq \gamma^*(\sigma)$ be such that any point $r \in \gamma^{ext}(\sigma, s)$ can be connected with the site t by a line in $\widehat{\mathbb{T}}_N$ avoiding $\widehat{I}^+(\sigma, s)$ (see (4.2)). Let $\widehat{J}^+(\sigma, s) \subset \widehat{\mathbb{T}}_N$ contain s and only its boundary is $\gamma^{ext}(\sigma, s)$. It is clear that $\widehat{J}^+(\sigma, s) \supset \widehat{I}^+(\sigma, s)$. The contour $\gamma^{ext}(\sigma, s)$ is called an external contour relatively to the site s .

Denote $\Gamma_{s,t} = \{\gamma^{ext}(\sigma, s) : \sigma \in \Omega_N^{s,t}\}$ the set of all external contours.

In what follows Ψ means either the block or the double-block. Recall that Ψ is a subgraph of (4.1) embedded in \mathbb{R}^2 . Further, for a given L_1 and L_2 we define a set $\mathfrak{P} = \{\Psi\}$ of permissible blocks which will participate in definition of a thick block-contour. First, all Λ -blocks belong to the set \mathfrak{P} . Second, as defined in (4.18), if L_1 is even, then all the horizontal $\Lambda_{h,1}^*$ - and $\Lambda_{h,2}^*$ -blocks belong to \mathfrak{P} ; and if L_2 is even, then all vertical $\Lambda_{v,1}^*$ - and $\Lambda_{v,2}^*$ -blocks belong to \mathfrak{P} as well.

Note that if L_1 and L_2 are odds, then the set \mathfrak{P} consists on only Λ -blocks. Note also that any double block Ψ from \mathfrak{P} consists of two disjoint Λ -blocks, say Λ' and Λ'' . We associate any such double block with a set $\Phi_\Psi \subset \Psi$ of edges connecting Λ' and Λ'' : $\Phi_\Psi = \{\langle u, v \rangle : u \in \Lambda' \text{ and } v \in \Lambda''\}$.

We say that a Λ -block Ψ from \mathfrak{P} intersects a contour γ^{ext} , $\Psi \cap \gamma^{ext} \neq \emptyset$, if there exists an edge $\langle u, v \rangle$, $u, v \in \Psi$ such that $\langle u^*, v^* \rangle \subset \gamma^{ext}$. We say that a Λ^* -block Ψ from \mathfrak{P} (if it exists) intersects a contour γ^{ext} , $\Psi \cap \gamma^{ext} \neq \emptyset$, if γ^{ext} intersects all edges from Φ_Ψ .

For any $\gamma^{ext} \in \Gamma_{s,t}$ we define $\mathcal{E} = \mathcal{E}(\gamma^{ext}) = \{\Psi : \Psi \cap \gamma^{ext} \neq \emptyset\}$. The set $\mathcal{E} = \mathcal{E}(\gamma^{ext})$ is called a *thick external contour* corresponding Peierls external contour γ^{ext} . Let $\mathcal{E}_b \subseteq \mathcal{E}$ be the subset of the Λ^* -blocks, and let $\mathcal{E}_0 = \mathcal{E} \setminus \mathcal{E}_b$ the set of Λ -blocks.

Denote $\Omega^*(\mathcal{E})$, the set of configurations σ , that generate the thick external contour \mathcal{E} . If $\sigma \in \Omega^*(\mathcal{E})$ then $\sigma \in \bigcap_{\Psi \in \mathcal{E}_0} \mathcal{B}(\sigma(\Psi)) \cap \bigcap_{\Psi^* \in \mathcal{E}_b} \mathcal{B}(\sigma(\Psi^*))$. Moreover $\sigma \in \bigcap_{\Psi \in \mathcal{E}_0} \mathcal{R}_\Psi \cap \bigcap_{\Psi^* \in \mathcal{E}_b} \mathcal{R}_{\Psi^*}$, that is

$$\Omega^*(\mathcal{E}) \subset \bigcap_{\Psi \in \mathcal{E}} \mathcal{R}_\Psi. \quad (4.62)$$

Next we estimate $\mu_{\beta,N}(\Omega^*(\mathcal{E}))$. First, we separate the set of Λ^* -blocks in all the possible Λ_k^* -blocks, $k \in \mathcal{D}$. Let $\mathcal{E}_b = \mathcal{E}_{h,1} \cup \mathcal{E}_{h,2} \cup \mathcal{E}_{v,1} \cup \mathcal{E}_{v,2}$, the subsets of Λ^* -blocks defined on the subgroups (4.19). Using (4.62), the Cauchy-Schwarz inequality, and the chessboard estimates (4.21) and (4.22), respectively, we obtain

$$\begin{aligned} \mu_{\beta,N}(\Omega^*(\mathcal{E})) &\leq \mu_{\beta,N}(\bigcap_{\Psi \in \mathcal{E}} \mathcal{R}_\Psi) \leq \sqrt{\mu_{\beta,N}(\bigcap_{\Psi \in \mathcal{E}_0} \mathcal{R}_\Psi)} \prod_{k \in \mathcal{D}} \sqrt[8]{\mu_{\beta,N}(\bigcap_{\Psi^* \in \mathcal{E}_k} \mathcal{R}_{\Psi^*})} \\ &\leq \prod_{\Psi \in \mathcal{E}_0} \left[\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{R}_\Psi)) \right]^{\frac{1}{2N^2}} \prod_{k \in \mathcal{D}} \prod_{\Psi^* \in \mathcal{E}_k} \left[\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}(\mathcal{R}_{\Psi^*})) \right]^{\frac{1}{4N^2}}. \end{aligned} \quad (4.63)$$

Recall that N^2 is the number of the Λ -blocks and $N^2/2$ is the number of the Λ^* -blocks. The probabilities $\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{R}_\Psi))$ when $\Psi \in \mathcal{E}_0$, and $\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}(\mathcal{R}_{\Psi^*}))$ when $\Psi^* \in \mathcal{E}_b$ do not depend on the position of Ψ (Ψ^*) thus we introduce a magnitude $\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{R}))$ meaning the probability of any propagated block. Moreover, we define the magnitude $\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}(\mathcal{R}^*))$ meaning the probability of any propagated double-block. Using (4.63) we obtain

$$\mu_{\beta,N}(\Omega^*(\mathcal{E})) \leq \left[\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N} \pi_{\mathbf{t}}(\mathcal{R})) \right]^{\frac{|\mathcal{E}_0|}{2N^2}} \prod_{k \in \mathcal{D}} \left[\mu_{\beta,N}(\bigcap_{\mathbf{t} \in \tilde{\mathbb{T}}_N^{(k)}} \pi_{\mathbf{t}}(\mathcal{R}^*)) \right]^{\frac{|\mathcal{E}_k|}{8(N^2/2)}}.$$

Using notation (4.23) and (4.26), and applying (4.30) and (4.31), we obtain

$$\begin{aligned} \mu_{\beta,N}(\Omega^*(\mathcal{E})) &\leq \mathfrak{z}_{\beta,N}(\mathcal{R})^{\frac{|\mathcal{E}_0|}{2}} \prod_{k \in \mathcal{D}} \mathfrak{z}_{\beta,N}(\mathcal{R}^*)^{\frac{|\mathcal{E}_k|}{8}} \\ &\leq 2^{B_1 B_2 \frac{|\mathcal{E}_0|}{2}} 4^{B_1 B_2 \frac{|\mathcal{E}_b|}{8}} \exp \left\{ -\frac{\beta}{8} |\mathcal{E}| \left(2J - \frac{hL_1 L_2}{L_1 + L_2} \right) \right\}. \end{aligned} \quad (4.64)$$

The next step is defining $\mathfrak{D}_{s,t} = \{\mathcal{E} = \mathcal{E}(\gamma^{ext}), \gamma^{s,t} \in \Gamma_{s,t}\}$. Next, we estimate $\mu_{\beta,N}(\Omega_N^{s,t})$ using the inclusion $\Omega_N^{s,t} \subset \bigcup_{\mathcal{E} \in \mathfrak{D}_{s,t}} \Omega^*(\mathcal{E})$. Considering the inequality (4.64), and a combinatorial argument, we obtain

$$\begin{aligned} \mu_{\beta,N}(\Omega_N^{s,t}) &\leq \sum_{\mathcal{E} \in \mathfrak{D}_{s,t}} \mu_{\beta,N}(\Omega^*(\mathcal{E})) \leq \sum_{n \geq 1} \sum_{\substack{n_0, n_b: \\ n_0 + n_b = n}} c^{n_0} c^{2n_b} (2^{n_0} \sqrt{2}^{n_b})^{B_1 B_2 / 2} e^{-\beta n \alpha / 8} \\ &= \sum_{n \geq 1} \left(c 2^{B_1 B_2 / 2} + c^2 4^{B_1 B_2 / 8} \right)^n e^{-\beta n \alpha / 8} \leq \sum_{n \geq 1} (c(c+1) 2^{B_1 B_2 / 2} e^{-\beta \alpha / 8})^n, \end{aligned} \quad (4.65)$$

where $\alpha = 2J - \frac{hL_1 L_2}{L_1 + L_2}$, $n_0 = |\mathcal{E}_0|$, $n_b = |\mathcal{E}_b|$, and c is a combinatorial constant related to the number of the thick contours. The way the constant c appears in (4.65) defined by our calculations the number of the thick contour having its length equal to n (see a justification below). If (4.33) holds true, then,

$$\mu_{\beta,N}(\Omega_N^{s,t}) \leq 2c(c+1) 2^{B_1 B_2 / 2} \exp \left\{ -\frac{\beta}{8} \left(2J - \frac{hL_1 L_2}{L_1 + L_2} \right) \right\}. \quad (4.66)$$

□

Justification of (4.65). The double-blocks create two neighboring vertices in the graph. Therefore the double-block is taking in account twice in (4.65). Thus the contribution of the double-block energy is estimated as $c^{2n_b} 4^{n_b B_1 B_2} \exp\{-\beta \alpha n_b\}$. □

5 2D ferromagnetic Ising model with alternating strips external field

In [23] it was studied a phase diagram of the 2D Ising model with alternating external field on 1D sublattices. In this section we prove the result of [23] about coexistence of two phases by using RP in a way similar to the considerations in previous sections. In fact we prove a more general result of the coexistence, including the coexistence result of [23]. The model in [23] is as follows. The external field is

$$h(s_1, s_2) = \begin{cases} h, & \text{if } s_2 \text{ is even,} \\ -h, & \text{if } s_2 \text{ is odd,} \end{cases} \quad (5.1)$$

where $h > 0$, and the Hamiltonian is defined by (2.3).

Thus this model is an “extreme” case of the cell-board model. Indeed, the external field in (5.1) can be obtained letting $L_1 = \infty$ and $L_2 = 1$. In [23], it is proved that a phase transition in this model holds true for β sufficiently large and $h < 2J - ke^{-\beta J}$, where k being a suitable positive constant. We propose a more general model, with $L_2 \geq 1$, for which we use the reflection positivity techniques to prove phase transition. We called this model, *the Ising model with alternating strips external field*.

Formally, consider $L \in \mathbb{N}$, for each integer n , we define a *strip* of size L by

$$F(n) = \{(t_1, t_2) \in \mathbb{Z}^2 : nL \leq t_2 < (n+1)L\}. \quad (5.2)$$

Note that $\mathbb{Z}^2 = \cup_{n \in \mathbb{Z}} F(n)$. In a similar way to (2.2), we think the strips being colored black or white. See Figure 2(b), where $L = 2$. Then, we define the subset \mathbf{Z}_+^* and \mathbf{Z}_-^* of \mathbb{Z}^2 .

$$\mathbf{Z}_+^* = \bigcup_{\substack{n: \\ n \text{ is even}}} F(n), \quad \mathbf{Z}_-^* = \mathbb{Z}^2 \setminus \mathbf{Z}_+^*. \quad (5.3)$$

Let $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$ the set of all configurations on \mathbb{Z}^2 . The formal Hamiltonian, for the Ising model with alternating strips, is defined by (2.3), and the external field is given by

$$h(s) = \begin{cases} h, & \text{if } s \in \mathbf{Z}_+^*, \\ -h, & \text{if } s \in \mathbf{Z}_-^*, \end{cases} \quad (5.4)$$

where $h > 0$. For this model, we obtain the following result.

Theorem 5.1. *Consider the Ising model defined by the Hamiltonian (2.3), with external field given by (5.4) and (5.3). If $h < 2J/L$, then there exists a suitable positive constant $k = k(L)$, such that for any $\beta > k/(2J - hL)$, there exist two distinct measures μ_β^+ and $\mu_\beta^- \in \mathcal{G}_\beta$, which satisfy*

$$\mu_\beta^\pm(\sigma(t) = \pm 1) > \frac{1}{2}. \quad (5.5)$$

Proof. We follow ideas of section 4.1. We remark that in this section, we shall use the same notation, without distinction with previous model.

First, we construct a torus \mathbb{T}_N by taking a subset T_N of \mathbb{Z}^2 of size $N \times NL$:

$$T_N = \{t = (t_1, t_2) \in \mathbb{Z}^2 : 0 \leq t_1 < N, 0 \leq t_2 < NL\},$$

where N is multiple of 4. Thus, \mathbb{T}_N is the factor-group $\mathbb{Z}/(N\mathbb{Z}) \times \mathbb{Z}/(NL\mathbb{Z})$. We consider the corresponding Hamiltonian with periodical boundary condition, as defined in (4.3).

Similar to (4.10), we define the sets of planes $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, given by

$$\begin{aligned} P_1^{(n)} &= \{t = (t_1, t_2) \in \mathbb{R}^2 : t_1 = n\}, \\ P_2^{(n)} &= \{t = (t_1, t_2) \in \mathbb{R}^2 : t_2 = nL + (L-1)/2\}, \end{aligned} \tag{5.6}$$

where $n < N$ a positive integer, and $\mathcal{P}_1 = \{P_1^{(n)}\}$, and $\mathcal{P}_2 = \{P_2^{(n)}\}$.

The set of planes \mathcal{P} , decompose the torus \mathbb{T}_N in rectangular blocks (see Figure 2(b)). It is easy to check (4.42) for all planes $P \in \mathcal{P}$, using same functions as in the proof of Proposition 4.3. Once the PR property is guaranteed, we apply the chessboard estimates (see (4.21)). For this reason, we define the Λ -blocks by translations of the block Λ . We construct Λ , considering on \mathbb{R}^2 the rectangle

$$\tilde{\Lambda} = \left\{ (t_1, t_2) : \left| t_1 + \frac{1}{2} \right| \leq \frac{1}{2}, \left| t_2 + \frac{1}{2} \right| \leq \frac{L}{2} \right\}.$$

Therefore, $\Lambda = \tilde{\Lambda} \cap \mathbb{Z}^2$. In other words, $\tilde{\mathbb{T}}_N = \{t = (t_1, t_2) \in \mathbb{T}_N : t_1 = n, t_2 = mL, n, m \in \mathbb{Z}\}$.

Now, we use the definition of bad-block event (see (4.27) for the definition of $R(\Lambda)$, and (4.28) for the event \mathcal{R}). Estimating $\mathfrak{z}_{\beta, N}(\mathcal{R})$, note that Lemma 4.6 still holds for this model. However, Lemma 4.7 should be re-written. Therefore, we will explain the steps to follow to obtain the desired estimate. From inequality (4.58), we consider $|\sum_{s \in T} h(s)| \leq hL$, for each $T \in \mathcal{T}_V$. That implies, an equivalent to (4.59). Remember that, $|\partial V| = |\partial^h V| + |\partial^v V| \geq 4$. Finally, we have

$$H_{[2 \times 2]}(\sigma^{V,+}) - H_{[2 \times 2]}(\sigma^+) \geq 2J|\partial V| - hL|\partial^v V| \geq 4(2J - hL). \tag{5.7}$$

Similar to (4.60), we say that $\mathfrak{z}_{\beta, N}(\mathcal{R}) \leq |R(\Lambda)| \exp(-\beta[2J - hL])$. We estimate $|R(\Lambda)| \leq 2^{2L'}$. In this case, $L' = L$, when L is even, and $L' = L + 1$, if L is odd.

Therefore, as in Proposition 4.5, for every $s, t \in \mathbb{T}_N$, the following inequality

$$\mu_{\beta, N}(\sigma(s) = +1, \sigma(t) = -1) \leq 2c(c+1)2^{2L'} \exp \left\{ -\frac{\beta}{2}(2J - hL) \right\}, \tag{5.8}$$

holds for all $\beta > k/(2J - hL)$. Here, the constant c is the same.

As a final step, we repeat considerations with conditional measures (4.37), where $t \in \mathbb{T}_N$, such that $t = s + (N/2, 0)$. Note that (4.34) still holds. Checking that, consider some configuration $\sigma \in \Omega_N$, such that $\sigma(\Lambda) \equiv +1$. So we can construct $\sigma' = -\theta_{Q_2}(\sigma) \in \Omega_N$, see equation (4.15). Then, $H_N(\sigma) = H_N(\sigma')$.

Finally, when $N \rightarrow \infty$, we obtain the infinite-volume Gibbs measures, μ_β^+ and μ_β^- . Therefore, by estimative to $\mathfrak{z}_{\beta, N}$ above and (5.8), there exists a constant $k > 0$, such that for all $\beta > k/(2J - hL)$, (5.5) holds true. \square

Acknowledgments: Manuel González Navarrete was supported by BecasChile, Comisión Nacional de Investigación Científica y Tecnológica. The research of Eugene Pechersky was carried out at the IITP-RAS at the expense of the Russian Foundation for Sciences (project No 14-50-00150). Anatoly Yambartsev thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq (grant 307110/2013-3) and Fundação de Amparo à Pesquisa do Estado de São Paulo, FAPESP (grant 2009/52379-8). The final publication is available at Springer via <http://dx.doi.org/10.1007/s10955-015-1392-9>

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